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Geometric Characterization of Totally Geodesic SODE Submanifolds

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ABSTRACT

Investigating the geometry of the tangent bundle (TM, π, M) over a smooth manifold M is one of the most significant fields of modern differential geometry and has remarkable applications in various problems specifically in the theory of physical fields. This significance provides a constructive setting for the development of novel notions and geometric structures such as systems of second order differential equations (SODE), metric structures, semisprays and nonlinear connections. Accordingly analysis of above mentioned concepts can be considered as a powerful tool for the thorough investigation of the geometric properties of a tangent bundle. This paper is devoted to exhaustive geometric analysis of totally geodesic SODE submanifolds. Investigating the induced SODE structure on submanifolds is our main objective in this paper. Indeed, it is demonstrated that the metric which is obtained from the metrizability of a given semispray, plays a fundamental role in inducing SODE structure on submanifolds. Particularly, a necessary and sufficient condition for an SODE submanifold to be totally geodesic is presented.

Keywords: SODE, semispray, dynamical covariant derivative, metrizability, nonlinear connection, totally geodesic submanifolds.

1. Introduction

Differential geometry of the total space of a manifold's tangent bundle has its origins in diverse fields of study such as Calculus of Variations, Differential Equations, Theoretical Physics and Mechanics. In recent years, it can be regarded as a distinguished domain of differential geometry and has noteworthy applications in specific problems from mathematical biology and mainly in the theory of physical fields Antonelli et al. (1993), Antonelli and Miron (1996), Beil (2003), Miron (1986), Miron and Anastasiei (1994, 1997). This significance provides a constructive setting for the development of novel notions and geometric structures such as systems of second order differential equations (SODE), metric structures, semisprays and nonlinear connections. Accordingly analysis of above mentioned concepts can be considered as a powerful tool for the thorough investigation of the geometric properties of a tangent bundle.

From the historical point of view, a principled investigation of the differential geometry of tangent bundles stated with Dombrowski (1962), Kobayashi and Nomizu (1969) and Yano and Ishihara (1973) in 1960's and 1970's. Specifically, Crampin (1971) and Grifone (1972) have considerably contributed to the geometry of the tangent bundle by introducing the notion of the nonlinear connection on the tangent bundle of a system of second order differential equations. In Miron (1986) the concept of generalized Lagrange spaces has been exhaustively introduced and investigated. Moreover, regarding covariant derivatives and geometric objects that can be associated to a system of second order differential equations, comprehensive researches have been fulfilled in Antonelli and Bucataru (2003), Crampin et al. (1996), Krupkova (1997), Lackey (1999), Sarlet (1982)(refer to Bucataru and Miron (2007) for more details).

In physical samples, regular Lagrangians, Finsler metrics and generalized Lagrange metrics induce various metric structures on TM which can be induced via distinct metric structures from Relativistic Optics or by Ehlers-Pirani-Schield axiomatic system. Metric geometry of TM associated to these metric structures has been exhaustively analyzed in Bucataru and Miron (2007). Among these geometries, in Bucataru and Miron (2007) the following specific aspects are mainly emphasized: The geometry of a Lagrange space can be constructed via the principles of Analytical Mechanics. Taking into account the fact that the geometry of a Finsler space is a special form of the Lagrange geometry, this geometry can be successfully constructed through the principles of Theoretical Mechanics. Since methods from Riemannian geometry are not sufficient for investigation of the geometry of generalized Lagrange spaces, one has to approach it as metric geometry on TM. Furthermore, geometric properties from Calculus of Variations can be thoroughly analyzed through the corresponding semispray and its differential geometry.

In recent years, the problem of metrizability has been investigated from several aspects. Indeed a semispray is called metrizable if the paths of the semispray are just the geodesics of some metric space. For a given arbitrary SODE one can associate a nonlinear connection and the corresponding dynamical covariant derivative. In this paper, we have applied the metric associated to a given SODE in order to induce a nonlinear connection on the submanifolds of the base manifold. Indeed, we have defined another SODE structure on the submanifolds via this nonlinear connection. Mainly, a significant geometric characterization of totally geodesic submanifolds is determined by inducing SODE structure on submanifolds.

2. Inverse Problem of the Calculus of Variations

A system of second order differential equations (SODE) on a configuration manifold M, whose associated coefficient functions do not depend explicitly on time, can be analyzed as a particular vector field on TM, which is called a semispray. In this section, we begin with a semispray S and analyze the induced geometric structures that will determine its corresponding geometric invariants. These geometric structures are defined by applying the nonlinear connection induced via a semispray. One of the significant tools in this section and in the whole paper is the dynamical covariant derivative induced by a semispray S. The dynamical covariant derivative we propose here, is associated to a given SODE and a nonlinear connection which is not fixed yet. In this article, we determine the nonlinear connection by requiring the compatibility of the dynamical covariant derivative with some corresponding geometric structures. Let M be a real n - dimensional smooth manifold and TM be its induced tangent bundle. The notion of semispray on the total space TM is related to the second order ordinary differential equation (SODE) on the base manifold M,

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0, \qquad i = 1, \cdots, n. \tag{1}$$

These equations on TM can be written as :

$$\frac{dy^i}{dt} + 2G^i(x, y) = 0, \quad y^i = \frac{dx^i}{dt}, \qquad i = 1, \cdots, n. \tag{2}$$

On the other hand, the equation (2) are the integral curves of the vector field

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}, \qquad i = 1, \cdots, n. \tag{3}$$

If S is a semispray with the coefficients $G^i(x,y)$, then the functions $G^i_j(x,y) = \frac{\partial G^i}{\partial y^j}$ are the coefficients of the nonlinear connection N.
The problem of compatibility between a system of second order differential equations and a metric structure on tangent bundle, has been studied by many authors and it is known as one of the Helmholtz conditions from the inverse problem of Lagrangian mechanic refer to Bucataru (2007), Bucataru and Miron (2007), Crasmareanu (2009), Sarlet (1982). The inverse problem of the calculus of variations can be formulated as follows: Under what conditions the solutions of an arbitrary autonomous system of second order differential equations (SODE) defined on an n - dimensional manifold M, are solutions of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad i \in \{1,2,\cdots,n\}, \tag{4}$$

for some Lagrangian function L. It is worth mentioning that throughout this paper we will not require the regularity of the Lagrangian functions and as a consequence, the given SODE (2) and the Euler-Lagrange equations (4) might not be equivalent. An approach to the inverse problem of the calculus of variations applies the Helmholtz conditions. These metrizability conditions are necessary and sufficient for the existence of a multiplier matrix $g_{ij}(x,\dot{x})$ such that

$$g_{ij}(x,\dot{x})\left(\frac{d^2x^j}{dt^2} + 2G^j(x,\dot{x})\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}, \quad i = 1,\cdots,n. \tag{5}$$

for some Lagrangian function $L(x,\dot{x})$.

Definition 2.1. *The dynamical covariant derivative associated to the semispray $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ is defined as follows:*

$$\nabla^d : \Gamma(VTM) \longrightarrow \Gamma(VTM)$$

$$\nabla^d(X^i \frac{\partial}{\partial y^i}) := (S(X^i) + G^i_j X^j)\frac{\partial}{\partial y^i}, \quad i,j = 1,\cdots,n.$$

∇^d satisfies the following conditions:
(a) : $\nabla^d(\frac{\partial}{\partial y^i}) = G^j_i \frac{\partial}{\partial y^j}$
(b) : $\nabla^d(X+Y) = \nabla^d X + \nabla^d Y$
(c) : $\nabla^d fX = S(f)X + f\nabla^d X$
for all $f \in C^\infty(TM)$ and $X,Y \in \Gamma(VTM)$.

Particularly, for each GL-metric g, we can define:

$$\nabla^d g(X,Y) = S(g(X,Y)) - g(\nabla^d X, Y) - g(X, \nabla^d Y) \tag{6}$$

Definition 2.2. *The semispray S is called metric with respect to metric g if $\nabla^d g = 0$, this is locally expressed as follows:*

$$S(g_{ij}) = g_{ik}G^k_j + g_{kj}G^k_i, \qquad i,j,k = 1,\cdots,n. \tag{7}$$

In this case, we call g the metric associated to the semispray S.

3. Characterization of Totally Geodesic Submanifolds via Inducing SODE Structure

3.1 Induced Nonlinear Connection

Let $\widetilde{M}^m$ be an immersed submanifold of M^{m+n} such that $\varphi : \widetilde{M} \longrightarrow M$ is an immersion and $\varphi(u) = (x^1(u), ..., x^{m+n}(u))$, where $u = (u^1, ..., u^m)$ and $x^i,\ i \in \{1, ..., m+n\}$ are smooth functions (differentiable of class C^∞). So we have:

$$\begin{gathered}\varphi_* : T\widetilde{M} \longrightarrow TM\\ (u^\alpha, v^\alpha) \longmapsto (x^i(u), y^i(u,v))\end{gathered}$$

and

$$\begin{aligned}&(a) : y^i(u,v) = B^i_\alpha v^\alpha \ ; \ (b) : B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} \ ;\\ &(c) : B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} \ ; \ (d) : B^i_{\alpha 0} = B^i_{\alpha\beta} v^\beta.\end{aligned} \tag{8}$$

Let S be a semispray which is locally represented as $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ and let $N = (G^i_j = \frac{\partial G^i}{\partial y^j})$ be the nonlinear connection associated to S. Assume that S is a metrizable semispray, due to Bucataru's definition, i.e. there exists a metric g, such that the following relation holds:

$$S(g_{ij}) = g_{ik}G^k_j + g_{kj}G^k_i. \tag{9}$$

Remark: Note that through out this section we mean by g, the metric which is obtained from the metrizability of the semispray S. Moreover, it is worth mentioning that throughout this section the ranges of the indices are as follows: $i,j,k \in \Big\{1,\cdots,m+n\Big\}$, $\alpha,\beta \in \Big\{1,\cdots,m\Big\}$ and $a,b \in \Big\{m+1,\cdots,m+n\Big\}$.

The metric $(g_{ij}(x))$ on TM induces a Riemannian metric $\widetilde{g}$ on $T\widetilde{M}$ such that

$$\widetilde{g}_{\alpha\beta}(u) = g_{ij}(x(u))B^{ij}_{\alpha\beta}, \qquad \alpha,\beta = 1,\cdots,m. \tag{10}$$

where $B^{ij}_{\alpha\beta} = B^i_\alpha B^j_\beta$.
Let $M' = TM - \{0\}$ and $\widetilde{M}' = T\widetilde{M} - \{0\}$ and $\{U, x^i, y^i\}$, $\{\widetilde{U}, u^\alpha, v^\alpha\}$ be coordinate systems on M' and $\widetilde{M}'$ respectively, and $\widetilde{U} = U \cap \varphi_*(\widetilde{M}')$, then the natural frame fields $\{\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial v^\alpha}\}$ and $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ on $\widetilde{M}'$ and M' resp. are related on $\widetilde{U}$ by Bejancu and Farran (2000):

$$\frac{\partial}{\partial u^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i} + B^i_{\alpha 0}\frac{\partial}{\partial y^i} \tag{11}$$

$$\frac{\partial}{\partial v^\alpha} = B^i_\alpha \frac{\partial}{\partial y^i} \tag{12}$$

where $B^i_{\alpha 0} = B^i_{\alpha\beta} v^\beta$.
Now respect to the metric g on VM', we consider the orthogonal complement of $V\widetilde{M}'$ as follows:

$$VM' = V\widetilde{M}' \oplus (V\widetilde{M}')^\perp \tag{13}$$

Let $\{B_a = B^i_a \frac{\partial}{\partial y^i}\}$ be a local field of orthonormal frames in $(V\widetilde{M}')^\perp$ with respect to g. We assume that B_a, $a = m+1, ..., m+n$ are spacelike. Hence, we have :

$$\begin{cases} (a): \ g(\dfrac{\partial}{\partial v^\alpha}, B_a) = 0, \\ (b): \ g(B_a, B_b) = \delta_{ab}. \qquad \alpha = 1,...,m \qquad a,b = m+1,...,m+n. \end{cases} \tag{14}$$

These relations are locally equivalent to:

$$\begin{cases} (a): \ g_{ij}B^i_\alpha B^j_a = g\Big(\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j}\Big)B^i_\alpha B^j_a = g\Big(B^i_\alpha \dfrac{\partial}{\partial y^i}, B^j_a \dfrac{\partial}{\partial y^j}\Big) \\ \qquad\qquad = g\Big(\dfrac{\partial}{\partial v^\alpha}, B_a\Big) = 0, \\ (b): \ g_{ij}B^i_a B^j_b = g\Big(\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j}\Big)B^i_a B^j_b = g\Big(B^i_a \dfrac{\partial}{\partial y^i}, B^j_b \dfrac{\partial}{\partial y^j}\Big) \\ \qquad\qquad = g\Big(B_a, B_b\Big) = \delta_{ab}. \\ \quad \alpha = 1,...,m \ , \quad a,b = m+1,...,m+n. \end{cases} \tag{15}$$

If $[B^i_\alpha B^i_a]$ is the transition matrix from natural frame fields $\{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^{m+n}}\}$ on VM' to frame fields $\{\frac{\partial}{\partial v^1}, ..., \frac{\partial}{\partial v^m}, B_{m+1}, ..., B_{m+n}\}$ adapted to (13), and if $[B^\alpha_i B^a_i]$ is the inverse of the matrix $[B^i_\alpha B^i_a]$, we have:

$$B^\alpha_i B^i_\beta = \delta^\alpha_\beta, \quad B^\alpha_i B^i_a = 0, \quad B^a_i B^i_\alpha = 0, \quad B^a_i B^i_b = \delta^a_b \tag{16}$$

$$B^i_\alpha B^\alpha_j + B^i_a B^a_j = \delta^i_j. \tag{17}$$

Now after contracting

$$\widetilde{g}_{\alpha\beta} = g_{ij}(x(u), y(u,v))B^{ij}_{\alpha\beta} \tag{18}$$

by $\widetilde{g}^{\beta\gamma} B^\alpha_k$ and taking into account (15.a) and (17) we have :

$$B^\gamma_k = g_{kj} B^j_\beta \widetilde{g}^{\beta\gamma} \tag{19}$$

After contracting (12) by B^α_j and by using (17), we deduce that (13) is Locally expressed by :

$$\frac{\partial}{\partial y^j} = B^\alpha_j \frac{\partial}{\partial v^\alpha} + B^a_j B_a \tag{20}$$

Finally, contracting (15.b) by B^b_k and due to (17) and (15.a) we infer:

$$B^a_i = g_{ij} B^j_b \delta^{ba} \tag{21}$$

also, we have :

$$\widetilde{g}^{\alpha\beta} = g^{ij} B^\alpha_i B^\beta_j \tag{22}$$

Consider the nonlinear connection $N = (G^i_j)$ on M'. According to Bejancu and Farran (2000), the nonlinear connection N, enables us to define an almost product structure on M' as follows. Consider a vector field X on M'. Then locally we have $X = X^i \frac{\delta}{\delta x^i} + \dot{X}^i \frac{\partial}{\partial y^i}$.

So, we define: $Q : \Gamma(TM') \longrightarrow \Gamma(TM')$ such that: $QX = \dot{X}^i \frac{\delta}{\delta x^i} + X^i \frac{\partial}{\partial y^i}$. We call Q the associate almost product structure to the nonlinear connection N. We denote by $\mathcal{F}(M')$ the algebra of smooth functions on M'. Then by using g, the metric associated to the semispray S and the projection morphisms v and h of TM' on VM' and HM' respectively, we define

$$G : \Gamma(TM') \times \Gamma(TM') \longrightarrow \mathcal{F}(M')$$

$$G(X, Y) = g(vX, vY) + g(QhX, QhY), \quad \forall\ X, Y \in \Gamma(TM').$$

We call G the Sasaki metric on M'.

Let G be the Sasaki metric on M'. Then on each coordinate neighborhood of $\widetilde{M}'$ we define the functions: $N^\beta_\alpha = \widetilde{g}^{\beta\gamma} G(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial v^\gamma})$.

Hence, we obtain a nonlinear connection $H\widetilde{M}' = (N_\alpha^\beta)$ on $\widetilde{M}'$. As a distribution, $H\widetilde{M}'$ is locally spanned by:

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - N_\alpha^\beta \frac{\partial}{\partial v^\beta} \tag{23}$$

We call $\widetilde{N} = (N_\alpha^\beta)$ the induced non-linear connection on $\widetilde{M}'$.

Lemma 3.1. *The induced nonlinear connection $\widetilde{N} = (N_\alpha^\beta)$ is locally expressed as follows:*

$$N_\alpha^\beta = B_i^\beta (B_{\alpha 0}^i + B_\alpha^j G_j^i) \tag{24}$$

Proof. Considering the Sasaki metric G on M', on each coordinate neighborhood of $\widetilde{M}'$ we have:

$$\begin{aligned}
N_\alpha^\beta &= \widetilde{g}^{\beta\gamma} G\Big(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial v^\gamma}\Big) = \widetilde{g}^{\beta\gamma} G\Big(B_\alpha^i \frac{\partial}{\partial x^i} + B_{\alpha 0}^i \frac{\partial}{\partial y^i}, B_\gamma^j \frac{\partial}{\partial y^j}\Big) \\
&= \widetilde{g}^{\beta\gamma} G\Big(B_\alpha^i (\frac{\delta}{\delta x^i} + G_i^k \frac{\partial}{\partial y^k}) + B_{\alpha 0}^i \frac{\partial}{\partial y^i}, B_\gamma^j \frac{\partial}{\partial y^j}\Big) \\
&= \widetilde{g}^{\beta\gamma} \Bigg(G\Big(B_\alpha^i \frac{\delta}{\delta x^i}, B_\gamma^j \frac{\partial}{\partial y^j}\Big) + G\Big(B_\alpha^i G_i^k \frac{\partial}{\partial y^k}, B_\gamma^j \frac{\partial}{\partial y^j}\Big) \\
&+ G\Big(B_{\alpha 0}^i \frac{\partial}{\partial y^i}, B_\gamma^j \frac{\partial}{\partial y^j}\Big)\Bigg) \\
&= \widetilde{g}^{\beta\gamma} \Bigg(G\Big(B_\alpha^i G_i^k \frac{\partial}{\partial y^k}, B_\gamma^j \frac{\partial}{\partial y^j}\Big) + G\Big(B_{\alpha 0}^i \frac{\partial}{\partial y^i}, B_\gamma^j \frac{\partial}{\partial y^j}\Big)\Bigg) \\
&= \widetilde{g}^{\beta\gamma} \Big(B_\alpha^i G_i^k B_\gamma^j g_{kj} + B_{\alpha 0}^i B_\gamma^j g_{ij} \Big)
\end{aligned}$$

Now by contracting (10) by $\widetilde{g}^{\beta\gamma} B_k^\alpha$ and taking into account (15) and (17), it is deduced that: $B_k^\gamma = g_{kj} B_\beta^j \widetilde{g}^{\beta\gamma}$. Consequently, we have:

$$N_\alpha^\beta = B_k^\beta G_i^k B_\alpha^i + B_{\alpha 0}^i B_i^\beta = B_i^\beta G_i^k B_\alpha^j + B_i^\beta B_{\alpha 0}^i = B_i^\beta (B_{\alpha 0}^i + B_\alpha^j G_j^i).$$

□

Moreover, by direct calculations we infer that :

$$dx^i = B_\alpha^i du^\alpha \tag{25}$$

$$dy^i = B_{\alpha 0}^i du^\alpha + B_\alpha^i dv^\alpha \tag{26}$$

We define 1- forms on $\widetilde{M}'$ as follows :

$$\delta v^{\alpha} = dv^{\alpha} + N^{\alpha}_{\beta} du^{\beta} \tag{27}$$

also, $\delta v^{\alpha} = B^{\alpha}_{j}\delta y^{j}$, where $\delta y^{j} = dy^{j} + G^{j}_{i}dx^{i}$.

Now, according to lemma 3.1. we can prove the following theorem:

Theorem 3.1. *As $HM' \oplus V\widetilde{M}'^{\perp}$ is orthogonal complementary vector bundle to $V\widetilde{M}'$ in TM' and $H\widetilde{M}'$ is orthogonal to $V\widetilde{M}'$, it is deduced that $H\widetilde{M}'$ is the vector subbundle of $HM' \oplus V\widetilde{M}'^{\perp}$. In other words, the local frame field $\left\{\frac{\delta}{\delta u^{\alpha}}\right\}_{\alpha=1}^{m}$ of the induced nonlinear connection $H\widetilde{M}' = \left(N^{\beta}_{\alpha}\right)$ is locally expressed by:*

$$\frac{\delta}{\delta u^{\alpha}} = B^{i}_{\alpha}\frac{\delta}{\delta x^{i}} + H^{a}_{\alpha}B_{a} \tag{28}$$

where $H^{a}_{\alpha} = B^{a}_{i}(B^{i}_{\alpha 0} + B^{j}_{\alpha}G^{i}_{j})$.

Proof. Taking into account relations (11), (12), (23) and identity (24) in lemma 3.1. we have:

$$\begin{aligned}\frac{\delta}{\delta u^{\alpha}} &= \frac{\partial}{\partial u^{\alpha}} - N^{\beta}_{\alpha}\frac{\partial}{\partial v^{\beta}}\\ &= B^{i}_{\alpha}\frac{\partial}{\partial x^{i}} + B^{i}_{\alpha 0}\frac{\partial}{\partial y^{i}} - B^{\beta}_{i}\left(B^{i}_{\alpha 0} + B^{j}_{\alpha}G^{i}_{j}\right)B^{k}_{\beta}\frac{\partial}{\partial y^{k}}\end{aligned} \tag{29}$$

Now, by inserting $\frac{\partial}{\partial x^{i}} = \frac{\delta}{\delta x^{i}} + G^{j}_{i}\frac{\partial}{\partial y^{j}}$ in relation (29) we have:

$$\frac{\delta}{\delta u^{\alpha}} = B^{i}_{\alpha}\frac{\delta}{\delta x^{i}} + B^{i}_{\alpha}G^{j}_{i}\frac{\partial}{\partial y^{j}} + B^{i}_{\alpha 0}\frac{\partial}{\partial y^{i}} - B^{\beta}_{j}B^{i}_{\alpha}G^{j}_{i}B^{k}_{\beta}\frac{\partial}{\partial y^{k}} - B^{\beta}_{i}B^{k}_{\beta}B^{i}_{\alpha 0}\frac{\partial}{\partial y^{k}} \tag{30}$$

Consequently, according to identities (16) and (17) it is deduced that:

$$\begin{aligned}\frac{\delta}{\delta u^{\alpha}} &= B^{i}_{\alpha}\frac{\delta}{\delta x^{i}} + B^{i}_{\alpha}G^{k}_{i}\frac{\partial}{\partial y^{k}} + B^{i}_{\alpha 0}\frac{\partial}{\partial y^{i}} - B^{i}_{\alpha}\delta^{k}_{j}G^{j}_{i}\frac{\partial}{\partial y^{k}} + B^{i}_{\alpha}B^{k}_{a}B^{a}_{j}G^{j}_{i}\frac{\partial}{\partial y^{k}}\\ &\quad -\delta^{k}_{i}B^{i}_{\alpha 0}\frac{\partial}{\partial y^{k}} + B^{k}_{a}B^{a}_{i}B^{i}_{\alpha 0}\frac{\partial}{\partial y^{k}}\\ &= B^{i}_{\alpha}\frac{\delta}{\delta x^{i}} + B^{a}_{i}\left(B^{i}_{\alpha 0} + B^{j}_{\alpha}G^{i}_{j}\right)B_{a}\\ &= B^{i}_{\alpha}\frac{\delta}{\delta x^{i}} + H^{a}_{\alpha}B_{a}.\end{aligned} \tag{31}$$

□

As we declared above, the metric associated to the semispray S has a fundamental role in defining the induced nonlinear connection on M'. We call $(\widetilde{M}, \widetilde{N} = (N^{\alpha}_{\beta}))$ the induced SODE submanifold of $(M, N = (G^i_j))$.

We have already defined the induced nonlinear connection $H\widetilde{M'} = (N^{\beta}_{\alpha})$ with local coefficients given by (24). Let ∇ be a linear connection on VM'. Hence, we proceed with the study of the geometric objects induced by ∇ on $\widetilde{M}$. Let $\widetilde{\nabla}$ be a linear connection on $V\widetilde{M'}$. We define the linear connection

$$\widetilde{\nabla} : \Gamma(T\widetilde{M'}) \times \Gamma(V\widetilde{M'}) \longrightarrow \Gamma(V\widetilde{M'})$$

such that:

$$\nabla_{\frac{\delta}{\delta u^{\beta}}} \frac{\partial}{\partial v^{\alpha}} = \widetilde{\nabla}_{\frac{\delta}{\delta u^{\beta}}} \frac{\partial}{\partial v^{\alpha}} + H^a_{\alpha\beta} B_a \tag{32}$$

$$\nabla_{\frac{\partial}{\partial v^{\alpha}}} \frac{\partial}{\partial v^{\beta}} = \widetilde{\nabla}_{\frac{\partial}{\partial v^{\beta}}} \frac{\partial}{\partial v^{\alpha}} + V^a_{\alpha\beta} B_a \tag{33}$$

$$\widetilde{\nabla}_{\frac{\delta}{\delta u^{\beta}}} \frac{\partial}{\partial v^{\alpha}} = \widetilde{F}^{\gamma}_{\alpha\beta} \frac{\partial}{\partial v^{\gamma}} \quad , \quad \widetilde{\nabla}_{\frac{\partial}{\partial v^{\beta}}} \frac{\partial}{\partial v^{\alpha}} = \widetilde{C}^{\gamma}_{\alpha\beta} \frac{\partial}{\partial v^{\gamma}} \tag{34}$$

If we consider the Berwald connection $(G^i_j\, , \nabla) = (G^i_j, G^i_{jk}, 0)$ on M, where G^i_j are the local coefficients of the canonical non-linear connection on M', then the local coefficients of the $\widetilde{\nabla}$ are given by

$$\widetilde{F}^{\gamma}_{\alpha\beta} = B^{\gamma}_k (B^k_{\alpha\beta} + G^k_{ij} B^{ij}_{\alpha\beta}) \quad , \quad \widetilde{C}^{\gamma}_{\alpha\beta} = 0 \tag{35}$$

$$H^a_{\alpha\beta} = B^a_k (B^k_{\alpha\beta} + G^k_{ij} B^{ij}_{\alpha\beta}) \quad , \quad V^a_{\alpha\beta} = 0 \tag{36}$$

Also, we have

$$\widetilde{F}^{\gamma}_{\alpha\epsilon} v^{\alpha} = \widetilde{F}^{\gamma}_{\epsilon\beta} v^{\beta} = N^{\gamma}_{\epsilon} \tag{37}$$

$$H^a_{\alpha\epsilon} v^{\alpha} = H^a_{\epsilon\beta} v^{\beta} = H^a_{\epsilon} \tag{38}$$

So, we can define the induced Berwald connection on the submanifold $\widetilde{M'}$.

3.2 Totally Geodesic Induced SODE Submanifolds

As we stated, the semispray $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, determines a nonlinear connection N with local coefficients $G^i_j = \frac{\partial G^i}{\partial y^j}$. We call N the nonlinear connection of the semispray S. Conversely, any nonlinear connection determines a semispray. Certainly the semispray of a nonlinear connection and the nonlinear connection of a semispray are different notions. By a simple computation, it can be shown that if G^i_j are the coefficients of the nonlinear connection N, then:

$$2G^{1i}(x,y) = G^i_j(x,y)y^j \tag{39}$$

G^{1i} is called the semispray of the nonlinear connection N. This means that in local coordinates the semispray of the nonlinear connection N with local coefficients G^i_j is given by:

$$S^1 = y^i \frac{\partial}{\partial x^i} - 2G^{1i}(x,y)\frac{\partial}{\partial y^i} \tag{40}$$

Similarly, for the induced nonlinear connection $\widetilde{N} = (N^\alpha_\beta)$, we can show that:

$$2N^{1\alpha}(u,v) = N^\alpha_\beta(u,v)v^\beta \tag{41}$$

So in local coordinates the semispray of $\widetilde{N} = (N^\alpha_\beta)$ is defined as follows:

$$\widetilde{S}^1 = v^\alpha \frac{\partial}{\partial u^\alpha} - 2N^{1\alpha}(u,v)\frac{\partial}{\partial v^\alpha} \tag{42}$$

Lemma 3.2. *Let S be semispray and N the nonlinear connection of S. Let S^1 be the semispray of N. Then the coefficient functions $G^i(x,y)$ are homogeneous of order two if and only if $S = S^1$.*

Proof. Let G^i be the local coefficients of the semispray S. Then N, the nonlinear connection of S has the local coefficients $G^i_j = \frac{\partial G^i}{\partial y^j}$. S^1 the semispray of N has the local coefficients $2G^{1i} = G^i_j y^j = (\frac{\partial G^i}{\partial y^j})y^j$. We have $S = S^1$ if and only if $G^i = G^{1i}$, this is equivalent to $2G^i = y^j(\frac{\partial G^i}{\partial y^j})$ which according to Euler's Theorem means that coefficient functions $G^i(x,y)$ are homogeneous of order two. □

Lemma 3.3. *Let $N = (G^k_j)$ be the nonlinear connection on M' and $\widetilde{N} = (N^\gamma_\alpha)$ be the induced nonlinear connection on $\widetilde{M}'$. Then the following relation holds:*

$$B^k_\gamma N^\gamma_\alpha + B^k_a H^a_\alpha = B^k_{\alpha 0} + B^j_\alpha G^k_j \tag{43}$$

Proof. We had the following relations:

$$(a): N^\gamma_\alpha = B^\gamma_i(B^i_{\alpha 0} + B^j_\alpha G^i_j) \qquad (b): H^a_\alpha = B^a_i(B^i_{\alpha 0} + B^j_\alpha G^i_j)$$

By contracting (a) and (b) by B^k_γ and B^k_a respectively, and then adding the results and due to (16) and (17), relation (43) will be obtained. □

Let $(\widetilde{M}, (N^\alpha_\beta))$ be the induced SODE submanifold of $(M, (G^i_j))$. Then $\widetilde{M}$ is said to be totally geodesic submanifold of M if any geodesic of $\widetilde{M}$ is a geodesic of M.

Theorem 3.2. *Let $(\widetilde{M}, \widetilde{N} = (N^\alpha_\beta))$ be the induced SODE submanifold of $(M, N = (G^i_j))$. Let $S^1 = y^i \frac{\partial}{\partial x^i} - 2G^{1i}(x,y)\frac{\partial}{\partial y^i}$ and $\widetilde{S}^1 = v^\alpha \frac{\partial}{\partial u^\alpha} - 2N^{1\alpha}(u,v)\frac{\partial}{\partial v^\alpha}$ be the semisprays of the nonlinear connections N and $\widetilde{N}$, respectively. Then $\widetilde{M}$ is a totally geodesic submanifold of M if and only if $H^a_0 = 0$.*

Proof. By contracting (43) with v^α and due to (8.d) we have:

$$B^k_\gamma N^\gamma_\alpha v^\alpha + B^k_a H^a_\alpha v^\alpha = B^k_{\alpha\gamma} v^\gamma v^\alpha + B^j_\alpha G^k_j v^\alpha \tag{44}$$

By setting $H^a_0 := H^a_\alpha v^\alpha$ and by using (8.a) and (41) we have:

$$2B^k_\gamma N^{1\gamma} + B^k_a H^a_0 = B^k_{\alpha\gamma} v^\gamma v^\alpha + y^j G^k_j \tag{45}$$

By setting $v^\gamma := \frac{du^\gamma}{dt}$, $v^\alpha := \frac{du^\alpha}{dt}$ and applying (39) we obtain:

$$2B^k_\gamma N^{1\gamma} + B^k_a H^a_0 = B^k_{\alpha\gamma} \frac{du^\gamma}{dt} . \frac{du^\alpha}{dt} + 2G^{1k} \tag{46}$$

By adding $B^k_\gamma \frac{d^2u^\gamma}{dt^2}$ to (46) we have:

$$B^k_\gamma(\frac{d^2u^\gamma}{dt^2} + 2N^{1\gamma}) + B^k_a H^a_0 = B^k_\gamma \frac{d^2u^\gamma}{dt^2} + B^k_{\alpha\gamma} \frac{du^\gamma}{dt} . \frac{du^\alpha}{dt} + 2G^{1k} \tag{47}$$

Since $\frac{dx^k}{dt} = B^k_\gamma \frac{du^\gamma}{dt}$, $\frac{d^2x^k}{dt^2} = B^k_{\alpha\gamma} \frac{du^\gamma}{dt}\frac{du^\alpha}{dt} + B^k_\gamma \frac{d^2u^\gamma}{dt^2}$, we will obtain the following relation:

$$B^k_\gamma(\frac{d^2u^\gamma}{dt^2} + 2N^{1\gamma}) + B^k_a H^a_0 = \frac{d^2x^k}{dt^2} + 2G^{1k}. \tag{48}$$

By lemma (3.3) and taking into account the relation (48), the proof will be completed. □

4. Concluding Remarks

Geometric analysis of the tangent bundle (TM, π, M) over a smooth manifold M can be regarded as one of the most significant field of modern differential geometry and has considerable applications in various problems specifically in the theory of physical fields. This importance provides a constructive setting for the progress of novel concepts and geometric structures such as systems of second order differential equations (SODE), metric structures, semisprays and nonlinear connections. Therefore, analysis of the notions declared above can be reckoned as a powerful tool for the exhaustive investigation of the geometric properties of a tangent bundle. In the last decades an increasing number of researches has been dedicated to the qualitative investigation of the solutions of systems of (non-)autonomous second (higher) order ordinary (partial) differential equations fields via some corresponding geometric structures. The notable fact regarding these entire investigations is the significant demand of a unifying geometric setting for a differential equation field considering the associated geometric structures and invariants.

In this paper, we have comprehensively analyzed the structure of totally geodesic SODE submanifolds via geometric point of view. Significantly, investigation of the induced SODE structure on submanifolds is our principal goal in current research. Recently, the problem of metrizability has been studied from several aspects. Indeed a semispray is called metrizable if the paths of the semispray are just the geodesics of some metric space. The problem of compatibility between a system of second order differential equations and a metric structure on tangent bundle, has been studied by many authors and it is known as one of the Helmholtz conditions from the inverse problem of Lagrangian mechanic. In this manuscript, it is mainly illustrated that the metric which is resulted from the metrizability of a given semispray, plays a fundamental role in inducing SODE structure on submanifolds. Specifically, we have presented a necessary and sufficient condition for an SODE submanifold to be totally geodesic.

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